The load due to pressure drop was $q = 0.5$psi. In addition, a peripheral seal load of 1.0 lb/in. was taken at the outer radius. Pressure and seal forces were assumed to be acting in the same direction in order to obtain conservative results (for worst possible operation conditions). In general, pressure and peripheral seal forces are opposite in direction. Deflections and stresses are shown for the center radial line of the sector. Maximum values along this line are the maximum values for the entire plate.

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# Boundary-Layer Flows with Swirl and Large Suction


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## Introduction

A NUMBER of devices such as swirl generators, swirl atomizers, rockets, vortex tubes, arc heaters, etc., involve flows with swirl. With these applications in mind, the effects of swirl, zero, or moderate mass transfer on the flow and heat transfer in the low-speed swirling laminar compressible boundary-layer flow over an axisymmetric surface with variable cross section have been investigated by Back[1] and Vimala[2] using the quasilinearization technique. This Note presents an analytical solution for the laminar swirling flow in a tube, a particular type of swirling flow (treated in Refs. 1 and 2) corresponding to a zero longitudinal acceleration parameter (i.e., $\bar{\beta}=0$), with large suction at the surface. Solutions are obtained by making use of the perturbation technique, as has been done by Nanbu.[3] Results compare well with those of Ref. 2, as the value of the suction parameter increases.


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## Analysis

The similarity equations governing the low-speed swirling flow of a perfect gas with constant specific heat, viscosity proportional to temperature, Prandtl number unity for the no-slip flow with longitudinal acceleration parameter $\bar{\beta}=0$ and with suction at the surface are[1,2]

$$f''' + ff'' + \bar{\alpha}[G(1-g_w) + g_w - G^2] = 0, \quad G'' + fG' = 0 \tag{1}$$

subject to boundary conditions

$$f(0) = f_w, f'(0) = G(0) = 0 \; f'(\infty) = G(\infty) = 1 \tag{2}$$

Here $f$ is the dimensionless stream function; $G$ stands for both the normalized swirl velocity and the enthalpy difference ratio; $\bar{\alpha}$ is the swirl parameter, and $g_w$ is the cooling parameter; $f_w = -\rho_w w_w (2\Xi)^{1/2}/r\rho_e\mu_e u_e$ is the suction parameter (where $-w_w$ is the suction velocity, and other symbols $\rho_w$, $u_e$, $\mu_e$, etc., are defined in Ref. 1); primes denote differentiation with respect to the independent similarity variable $Z$. Defining Ref. 3:

$$\bar{Z} = f_w Z, \; f(Z) = f_w F(\bar{Z}), \; G(Z) = G(\bar{Z}) \tag{3}$$

and substituting in Eqs. (1) and (2), we have

$$F''' + FF'' + \epsilon_2[G(1-g_w) + g_w - G^2] = 0 \tag{4a}$$

$$G'' + F G' = 0 \tag{4b}$$

with

$$F(0) = G(\infty) = 1, \quad F'(0) = G(0) = 0, \quad F'(\infty) = \epsilon_1 \tag{5}$$

where

$$\epsilon_1 = f_w^{-2}, \qquad \epsilon_2 = \bar{\alpha} f_w^{-4} \tag{6}$$

and primes denote differentiation with respect to $\bar{Z}$.

For large suction, $f_w$ will assume large positive values so that $\epsilon_1$ and $\epsilon_2$ are small. In that case, $F$ and $G$ can be expanded in terms of the small perturbation quantities $\epsilon_1$ and $\epsilon_2$ as follows[4]

$$F = F_o + \epsilon_1 F_{11} + \epsilon_2 F_{12} + \epsilon_1^2 F_{21} + \epsilon_2^2 F_{22} + \ldots \tag{7a}$$

$$G = G_o + \epsilon_1 G_{11} + \epsilon_2 G_{12} + \epsilon_1^2 G_{21} + \epsilon_2^2 G_{22} + \ldots \tag{7b}$$

Substitution for $F$ and $G$, given by Eq. (7), in Eqs. (4) and (5) yields the following sets of differential equations and boundary conditions for $F_o$, $G_o$, and $F_{ij}$, $G_{ij}$ $(i,j=1,2)$:

Zeroth order $0(1)$

$$F_o''' + F_o F_o'' = 0; \quad F_o(0) = 1, \quad F_o'(0) = F_o'(\infty) = 0 \tag{8a}$$

$$G_o'' + F_o G_o' = 0; \quad G_o(0) = 0, \quad G_o(\infty) = 1 \tag{8b}$$

First order $0(\epsilon_1)$

$$F_{11}''' + F_o F_{11}'' + F_{11} F_o'' = 0; \quad F_{11}(0) = F_{11}'(0) = 0, \quad F_{11}'(\infty) = 1 \tag{9a}$$

$$G_{11}'' + F_o G_{11}' + F_{11} G_o' = 0; \quad G_{11}(0) = G_{11}(\infty) = 0 \tag{9b}$$

First order $0(\epsilon_2)$

$$F_{12}''' + F_o F_{12}'' + F_{12} F_o'' + G_o(1-g_w) + g_w - G_o^2 = 0; \tag{10a}$$

$$G_{12}'' + F_{12} G_o' + F_o G_{12}' = 0; \tag{10b}$$

$$F_{12}(0) = F'_{12}(0) = F_{12}'(\infty) = G_{12}(0) = G_{12}(\infty) = 0 \tag{10c}$$

Second order $0(\epsilon_1^2)$

$$F'''_{21}+F_oF''_{21}+F_{21}F''_o+F_{11}F''_{11}=0 \qquad (11a)$$

$$G''_{21}+F_oG'_{21}+F_{21}G'_o+F_{11}G'_{11}=0 \qquad (11b)$$

$$F_{21}(0)=F'_{21}(0)=F'_{21}(\infty)=G_{21}(0)=G_{21}(\infty)=0 \qquad (11c)$$

Second order $0(\epsilon_2^2)$

$$F'''_{22}+F_oF''_{22}+F_{22}F''_o+F_{12}F''_{12}+G_{12}(1-g_w)-2G_oG_{12}=0; \qquad (12a)$$

$$G''_{22}+F_oG'_{22}+F_{22}G'_o+F_{12}G'_{12}=0; \qquad (12b)$$

$$F_{22}(0)=F'_{22}(0)=F'_{22}(\infty)=G_{22}(0)=G_{22}(\infty)=0 \qquad (12c)$$

The solutions of the previous sets of equations are given by

$$F_o=1 \qquad (13a)$$

$$F_{11}=e^{-\bar{Z}}+\bar{Z}-1 \qquad (13b)$$

$$F_{12}=(3/4+g_w)-(1/2+g_w)e^{-\bar{Z}} - (1/4)\ e^{-2\bar{Z}}-(1+g_w)\bar{Z}e^{-\bar{Z}} \qquad (13c)$$

$$F_{21}=5/4-(3/2)e^{-\bar{Z}}+(1/4)e^{-2\bar{Z}} -\bar{Z}e^{-\bar{Z}}-(1/2)\bar{Z}^2e^{-\bar{Z}} \qquad (13d)$$

$$\begin{aligned}F_{22}=&-(1/432)[512+1349\ g_w+891\ g_w^2]\\ &+(1/144)[99+286\ g_w+216\ g_w^2]e^{-\bar{Z}}\\ &+(1/48)[11+39\ g_w+27g_w^2]e^{-2\bar{Z}}\\ &+(1/432)[113+140\ g_w]e^{-3\bar{Z}}+(1/144)e^{-4\bar{Z}}\\ &+(1/24)[20+59\ g_w+42\ g_w^2]\bar{Z}e^{-\bar{Z}}\\ &+(1/8)[8+16g_w+7g_w^2]\bar{Z}e^{-2\bar{Z}}\\ &+(1/8)\ (1+g_w)\bar{Z}e^{-3\bar{Z}}\\ &+(1/4)[3+7g_w+4g_w^2]\bar{Z}^2e^{-\bar{Z}}\\ &+(1/4)[1+2g_w+g_w^2]\bar{Z}^2e^{-2\bar{Z}}\end{aligned} \qquad (13e)$$

$$G_o=1-e^{-\bar{Z}} \qquad (14a)$$

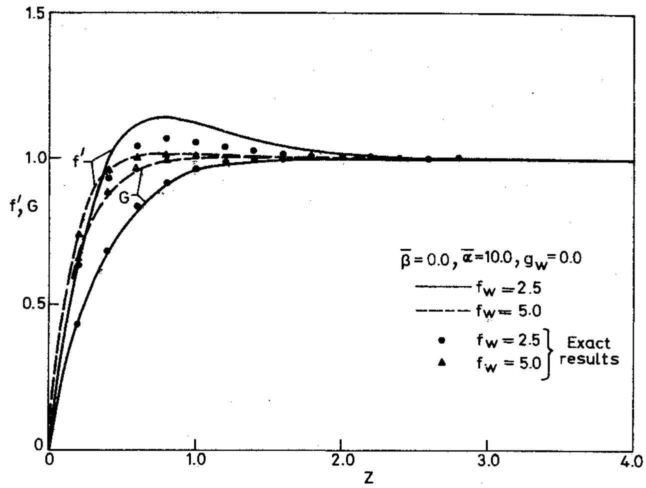


**Fig. 1 Comparison of the approximate analytical solution for large suction with exact results: velocity and enthalpy profiles.**

$$G_{11}=(1/2)[e^{-\bar{Z}}-e^{-2\bar{Z}}+\bar{Z}^2e^{-\bar{Z}}] \qquad (14b)$$

$$\begin{aligned}G_{12}=&-(5/4)[5/6+g_w]e^{-\bar{Z}}\\ &+[1+(5/4)g_w]e^{-2\bar{Z}}+(1/24)e^{-3\bar{Z}}\\ &+[3/4+g_w]\bar{Z}e^{-\bar{Z}}+(1/2)[1+g_w]\bar{Z}e^{-2\bar{Z}}\end{aligned} \qquad (14c)$$

$$\begin{aligned}G_{21}=&-(55/24)e^{-\bar{Z}}+(5/2)e^{-2\bar{Z}}-(5/24)e^{-3\bar{Z}}\\ &+(1/4)\bar{Z}e^{-\bar{Z}}+\bar{Z}e^{-2\bar{Z}}-(3/4)\bar{Z}^2e^{-\bar{Z}}+(1/2)\bar{Z}^2e^{-2\bar{Z}}\\ &-(1/8)\bar{Z}^4e^{-\bar{Z}}\end{aligned} \qquad (14d)$$

$$\begin{aligned}G_{22}=&(1/51840)[92189+237680g_w+152160\ g_w^2]e^{-\bar{Z}}\\ &-(1/144)[138+374\ g_w+252\ g_w^2]e^{-2\bar{Z}}\\ &-(1/1728)[1271+3256\ g_w+2048\ g_w^2]e^{-3\bar{Z}}\\ &-(1/1296)[107+134\ g_w]e^{-4\bar{Z}}-(11/5760)e^{-5\bar{Z}}\\ &-(1/864)[349+988\ g_w+702\ g_w^2]\bar{Z}e^{-\bar{Z}}\\ &-(1/24)[38+101g_w+66\ g_w^2]\bar{Z}e^{-2\bar{Z}}\\ &-(1/288)[253+548\ g_w+286g_w^2]\bar{Z}e^{-3\bar{Z}}\end{aligned} \qquad (14e)$$

**Table 1 Skin friction and heat transfer parameters**

| | | | $f''_w$ | | $G'_w$ | |
|---|---|---|---|---|---|---|
| $\bar{\alpha}$ | $g_w$ | $f_w$ | Exact[2] | Approximate (present) | Exact[2] | Approximate (present) |
| | | 2.5 | 2.6666 | 2.7000 | 2.6666 | 2.6467 |
| | | 3.0 | 3.1452 | 3.1667 | 3.1452 | 3.1358 |
| | 0 | 3.5 | 3.6283 | 3.6429 | 3.6283 | 3.6234 |
| 0 | and | 4.0 | 4.1147 | 4.1250 | 4.1147 | 4.1120 |
| | 1 | 4.5 | 4.6037 | 4.6111 | 4.6037 | 4.6020 |
| | | 5.0 | 5.0945 | 5.1000 | 5.0945 | 5.0933 |
| | | 2.5 | 4.1299 | 4.3444 | 2.7154 | 2.7168 |
| | | 3.0 | 4.4657 | 4.6904 | 3.1795 | 3.1873 |
| 10 | 0 | 3.5 | 4.8228 | 5.0053 | 3.6530 | 3.6588 |
| | | 4.0 | 5.1999 | 5.3411 | 4.1330 | 4.1367 |
| | | 4.5 | 5.5945 | 5.7034 | 4.6174 | 4.6197 |
| | | 5.0 | 6.0038 | 6.0889 | 5.1050 | 5.1064 |
| | | 2.5 | 7.0587 | 6.5382 | 2.7797 | 2.7039 |
| | | 3.0 | 7.1265 | 7.2979 | 3.2272 | 3.2317 |
| 10 | 1 | 3.5 | 7.2395 | 7.5266 | 3.6884 | 3.7007 |
| | | 4.0 | 7.4000 | 7.6688 | 4.1596 | 4.1693 |
| | | 4.5 | 7.6057 | 7.8300 | 4.6376 | 4.6443 |
| | | 5.0 | 7.8507 | 8.0324 | 5.1206 | 5.1250 |

$$-(1/24)(1+g_w)\bar{Z}e^{-4\bar{Z}}$$
$$-(1/32)[9+24g_w+16g_w^2]\bar{Z}^2e^{-\bar{Z}}$$
$$-(1/4)[3+7g_w+4g_w^2]\bar{Z}^2e^{-2\bar{Z}}$$
$$-(5/24)[1+2g_w+g_w^2]\bar{Z}^2e^{-3\bar{Z}} \quad (14e)$$

From Eqs. (3, 7, 13 and 14), the skin friction and heat transfer parameters $f''_w$ and $G'_w$ are given analytically by

$$f''_w=f_w+(1/2f_w)[1+\bar{\alpha}(1+2g_w)]$$
$$-(\bar{\alpha}^2/72f_w^5)[25+73g_w+54g_w^2]+\ldots \quad (15)$$

$$G'_w=f_w+(1/2f_w+1/12f_w^3)[\bar{\alpha}(2+3g_w)-10]$$
$$-(\bar{\alpha}^2/2160f_w^7)[481+1340g_w+940g_w^2]+\ldots \quad (16)$$

Further, the velocity and enthalpy can also be calculated from the expressions

$$f'=f_w^2[\epsilon_1F'_{11}+\epsilon_2F'_{12}+\epsilon_1^2F'_{21}+\epsilon_2^2F'_{22}] \quad (17a)$$

$$G=[G_o+\epsilon_1G_{11}+\epsilon_2G_{12}+\epsilon_1^2G_{21}+\epsilon_2^2G_{22}] \quad (17b)$$

where

$$F'_{11}=1-e^{-\bar{Z}}, \quad (18a)$$

$$F'_{12}=-(1/2)e^{-\bar{Z}}+(1/2)e^{-2\bar{Z}}+(1+g_w)\bar{Z}e^{-\bar{Z}}, \quad (18b)$$

$$F'_{21}=(1/2)[e^{-\bar{Z}}-e^{-2\bar{Z}}+\bar{Z}^2e^{-\bar{Z}}], \quad (18c)$$

$$F'_{22}=(1/144)[21+68g_w+36g_w^2]e^{-\bar{Z}}$$
$$+(1/24)[13+9g_w-6g_w^2]e^{-2\bar{Z}}$$
$$-(1/144)[95+122g_w]e^{-3\bar{Z}}$$
$$-(1/36)e^{-4\bar{Z}}+(1/24)[16+25g_w+6g_w^2]\bar{Z}e^{-\bar{Z}}$$
$$-(1/4)[6+12g_w+5g_w^2]\bar{Z}e^{-2\bar{Z}}$$
$$-(3/8)[1+g_w]\bar{Z}e^{-3\bar{Z}}$$
$$-(1/4)[3+7g_w+4g_w^2]\bar{Z}^2e^{-\bar{Z}}$$
$$-(1/2)[1+2g_w+g_w^2]\bar{Z}^2e^{-2\bar{Z}} \quad (18d)$$

## Results and Conclusions

A comparison of the velocity and enthalpy profiles obtained from the above analytical solutions for $\bar{\alpha}=10.0$, $\bar{\beta}=0.0$, $g_w=0.0$, $f_w=2.5$, 5.0 with the numerical results obtained by quasilinearization[2] is made in Fig. 1. The critical wall parameters $f''_w$ and $G'_w$ calculated using Eqs. (15) and (16) are presented in Table 1, as compared with their values[2] obtained numerically. It is clear from the table as well as the figure that there is good agreement for large values of the suction parameter $f_w$. It might also be noted that, for $f_w=5.0$, the longitudinal velocity overshoot is very small.

Thus, the present investigation supplies analytical solution for the large suction case. Further, it reveals the fact that with very large rates of suction, the velocity overshoot can be almost eliminated even in flows with swirl characterized by velocity overshoot in the longitudinal direction.

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# Stability Predictions for Combustors with Acoustic Absorbers and Continuous Combustion Distributions

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## Introduction

In an earlier paper[1] the authors presented an analytical technique suitable for determining the stability of combustors with partial length liners, and with an arbitrary number of planar sources of mass and energy distributed along the combustion chamber axis. This type of combustor model was chosen to approximate the actual continuous combustion distributions found in liquid rocket combustors. The approach led to a system of partial differential equations for the source free flow between the planar sources which could be solved under oscillatory conditions to determine the response of the planar sources necessary to sustain periodic oscillations. This type of analysis was chosen because a solution of the partial differential equations for the flowfield including mass, momentum, and energy sources was not available at that time.

This paper presents results of calculations performed in which the source terms are included in the analysis, and the realistic case of a continuous distribution of combustion sources in the axial direction is considered. These results exhibit a qualitative similarity to the multi-zone work. However, there is a substantial upward shift of the curves in all cases relative to the curves obtained in the earlier approximate analysis indicating a marked increase in predicted chamber stability. This difference in stability predictions between the two models is traced to some important damping effects associated with the distributed source terms which are neglected in the multi-zone model.

## Analysis

If it is assumed (consistent with the earlier work[1]) that the combustion product gas is a single component calorically perfect inviscid and nonheat conducting gas, and that drag forces on the liquid propellant droplets and heat transfer between droplets and gas are negligible, then the conservation equations for the three-dimensional flowfield in the chamber may be written in the following nondimensional form:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho q=Q \quad \text{(continuity)}$$

$$\rho\frac{Dq}{Dt}-\frac{1}{\gamma}\nabla p=Q(q_\ell-q) \quad \text{(momentum)}$$

$$\frac{D\sigma}{Dt}=\frac{Q}{p}\left[h_\ell-h-\frac{\gamma-1}{2}(2q\cdot q_\ell-q^2)\right] \quad \text{(entropy)} \quad (1)$$

The associated state equations are $p=\rho T$, $\rho=p^{1/\gamma}e^{-\sigma}$, and $T=h$. In these equations pressure, density, enthalpy, and temperature ($p$, $\rho$, $h$, $T$) are nondimensionalized through division by their steady-state values at the injector. $Q$ represents the nondimensional volumetric mass source and is normalized by division by $(\bar{\rho}_o^*\bar{a}_o^*/R^*)$. The velocities $q$ and $q_\ell$ are made nondimensional through division by $\bar{a}_o^*$. The non-

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